Trends in Mathematics

Julien Barral
Stéphane Seuret
Editors

Further Developments in Fractals and Related Fields

Mathematical Foundations and Connections

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Mathematical Foundations and Connections

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In memoriam

Benoît Mandelbrot,

whose friendship

was precious to us,

and whose scientific legacy

is so important to our community.

Preface

This volume is a collection of 13 peer-reviewed chapters consisting of expository/survey chapters and research articles on fractals. Many of these chapters were presented at the second edition of the international conference “Fractals and Related Fields,” held on Porquerolles Island, France, in June 2011. The success of this event proved the dynamism of the mathematical activity in the numerous branches connected to fractal geometry.

The selected chapters cover the following topics:

- Geometric measure theory
- Ergodic theory, dynamical systems
- Harmonic analysis
- Multifractal analysis
- Number theory
- Probability theory

The three surveys are written by famous experts in their respective fields. The other chapters are either original contributions or accessible expositions of very recent developments, also written by leaders in their respective domains.

This book naturally follows the previous one, “Recent Development in Fractals and Related Fields” which was published after the first conference, “Fractals and Related Fields.” It is intended for researchers and graduate students wishing to discover new trends in fractal geometry.

Villetaneuse, France — Julien Barral
Créteil Cedex, France — Stéphane Seuret

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The Rauzy Gasket

Pierre Arnoux and Štěpán Starosta

Abstract We define the Rauzy gasket as a subset of the standard two-dimensional simplex associated with letter frequencies of ternary episturmian words. We prove that the Rauzy gasket is homeomorphic to the usual Sierpiński gasket (by a two-dimensional generalization of the Minkowski ? function) and to the Apollonian gasket (by a map which is smooth on the boundary of the simplex). We prove that it is also homothetic to the invariant set of the fully subtractive algorithm, hence of measure 0.

1 Introduction

Strict episturmian ternary words, also called Arnoux–Rauzy words, are a natural generalization of Sturmian words (see Sect. 2 for the definitions). Each such word is uniquely ergodic, and in particular, its letters have a well-defined frequency; one can prove that these frequencies completely define the minimal symbolic system associated with such a word.

These dynamical systems are associated with a particular family of interval exchange transformations (see [1]). It is known that some of these systems (in particular those defined by a substitution) can be represented by a toral rotation,

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