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Stabilization of Navier–Stokes Flows

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Preface

In the last years, notable progresses were obtained in mathematical theory of stabilization of equilibrium solution to Newtonian fluid flows as a principal tool to eliminate or attenuate the turbulence. One of the main results obtained in this direction is that the equilibrium solutions to Navier–Stokes equations are exponentially stabilizable by finite-dimensional feedback controllers with support in the interior of the domain or on the boundary. This book was completed in the idea to present these new results and techniques which are in our opinion the core of a discipline still in development and from which one might expect in the future some spectacular achievements.

Beside internal and boundary stabilization of Navier–Stokes equations, the stochastic stabilization and robustness of stabilizable feedbacks are also discussed. We had in mind a rigorous mathematical treatment of the stabilization problem, which relies on some advanced results and techniques involving the theory of Navier–Stokes equations and functional analysis as well. We tried to answer to the following questions: which is the structure of the stabilizing feedback controller and how can be designed by using a minimal set of eigenfunctions of the Stokes–Oseen operator. Though most of the feedback controllers constructed here are conceptual and their practical implementation requires a computational effort which still remains to be done, the analysis developed here provides a rigorous pattern for the design of efficient stabilizable feedback controllers in most specific problems of practical interest. To this purpose, the exposition is in mathematical style: definitions, hypotheses, theorems, proof. It should be emphasized that no rigorous stabilization theory with internal or boundary controllers is possible without unique continuation theory for the solutions to Stokes–Oseen equations.

By including a preparatory chapter on infinite-dimensional differential equations and a few appendices pertaining unique continuation properties of eigenfunctions of the Stokes–Oseen operator and stochastic processes, we have attempted to make this work essentially self-contained and so, accessible to a broad spectrum of readers. What is assumed of the reader is a knowledge of basic results in linear functional analysis, linear algebra, probability theory and general variational theory of elliptic, parabolic and Navier–Stokes equations, most of these being reviewed in Chap. 1 and in Sects. 3.8 and 4.5. An important part of the material included in this book represent the personal contribution of the author and his coworkers and, though we mentioned the basic references and a brief presentation of other significant works in this field, we did not present them, however, in details. In fact, the presentation was confined to the stabilization techniques based on the spectral decomposition of the linearized system in stable and unstable systems and so we have omitted other important results in literature.

The author is indebted to Cătălin Lefter who made pertinent observations and suggestions. I also thank Irena Lasiecka and Roberto Triggiani for useful discussions on several results presented in this book. Also, I am indebted to Mrs. Elena Mocanu from Institute of Mathematics in Iaşi who prepared this text for printing.

I wish to express my thanks to Professor Miroslav Krstic, from University of California, San Diego, for the invitation to write this book for the Springer series *Communication and Control Engineering* he is coordinating.

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Contents

Symbols and Notation

R^d	the d-dimensional Euclidean space
R	the real line $(-\infty, +\infty)$
R^+	the half line $[0, +\infty)$
$\mathbb{C}$	the complex space
$\mathbb{C}^d$	the d-dimensional complex space
$x \cdot y$	the dot product of vectors $x, y \in R^d$
$\vert \cdot \vert_X, \ \Vert \cdot \Vert_X$	the norm of a linear normed space X
$D_i y$	$\frac{\partial y}{\partial x_i}, \ i = 1, \dots, d$
∇f	the gradient of the map $f : X \to Y$
$\nabla \cdot f$	the divergence of vector field $f : \mathcal{O} \to R^d \subset R^d$
$L(X, Y)$	the space of linear continuous operators from X to Y
$\Vert \cdot \Vert_{L(X,Y)}$	the norm of $L(X, Y)$
$X^*, \ X'$	the dual of the space X
$(x, y), \ (x, y)_H$	the scalar product of the vectors $x, y \in H$ (a Hilbert space). If $x \in X, \ y \in X^*$, this is the value of y at x
A^α	the fractional power of order $\alpha \in (0, 1)$ of the operator $A : D(A) \subset H \to H$
$D(A)$	the domain of the operator A
$A^*, \ A'$	the adjoint of the operator A
A^{-1}	the inverse of the operator A
$\{\Omega, \mathcal{F}, \mathbb{P}\}$	the probability space
$L^p(0, T; X)$	(X Banach space) the space of all p-sumable functions from $[0, T]$ to X
$L^p_{\text{loc}}(0, \infty; X)$	the space of measurable functions $y : (0, \infty) \to X$ which are p-integrable on each interval $(a, b) \subset (0, \infty)$
$y'(t), \frac{dy}{dt}(t)$	the derivative of the function $y : [0, T] \to X$
$AC([0, T]; X)$	the space of absolutely continuous functions from $[0, T]$ to X
$W^{1,p}([0, T]; X)$	the space $\{y \in AC([0, T]; \ y' \in L^p(0, T; X)\}$
$C([0, T]; X)$	the space of all continuous functions from $[0, T]$ to X
$C_w([0, T]; X)$	the space of all weakly continuous functions from $[0, T]$ to X
e^{At}	the C_0-semigroup generated by A
$H^k(\mathcal{O})$	the Sobolev space of order k on $\mathcal{O} \subset R^d$

Chapter 1
Preliminaries

The purpose of this chapter is to briefly present without proofs, for later use, certain notions and fundamental results pertaining linear operators in Banach spaces, boundary value problems, nonlinear dynamics in Hilbert spaces and existence theory of Navier–Stokes equations.

1.1 Banach Spaces and Linear Operators

A Banach space is a linear normed space which is complete. The norm of the Banach space X (real or complex) is denoted by $\|\cdot\|_X$ and $L(X,X)$ is the space of all linear continuous operators from X to itself. If X is a real Banach space (that is, over the real field R), then its complexification $\widetilde{X}$ is the space $\widetilde{X} = X + iX$, that is, $\widetilde{X} = \{x + iy,\ x, y \in X\}$ with the norm $\|x + iy\| = \|x\|_X + \|y\|_X$.

If A is a linear operator from X to Y, we denote by $D(A)$ its domain, that is, $D(A) = \{x \in X;\ Ax \neq \emptyset\}$ and by $R(A)$ its range, that is, $R(A) = \{y \in Y;\ y = Ax,\ x \in D(A)\}$. The linear operator is said to be *closed* if its graph $\{(x, y) \in X \times Y;\ y = Ax\}$ is closed, that is, if $x_n \xrightarrow{X} x$ and $y_n \in Ax_n \xrightarrow{Y} y$ implies that $y = Ax$. Here we use the symbol $\xrightarrow{X}$ for the convergence in the norm $\|\cdot\|_X$, that is, the strong convergence. The linear operator A is said to be densely defined if its domain $D(A)$ is dense in X. The inverse of A is denoted A^{-1}.

For each $\lambda \in \mathbb{C}$ (the complex field) denote by $(\lambda I - A)^{-1} \in L(X, X)$ the resolvent of $A : D(A) \subset X \to X$ and by $\rho(A)$ the resolvent set, $\rho(A) = \{\lambda \in \mathbb{C};\ (\lambda I - A)^{-1} \in L(X, X)\}$ and by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of A. In each component of $\rho(A)$ the function $\lambda \to (\lambda I - A)^{-1}$ is holomorphic.

The number $\lambda \in \mathbb{C}$ is said to be *eigenvalue* of the linear operator $A : D(A) \subset X \to X$ if there is $x \in D(A)$, $x \neq 0$, such that $Ax = \lambda x$.

The corresponding vectors x are called *eigenvectors*. If λ is eigenvalue for A, then the dimension of the linear eigenvector space $\mathrm{Ker}(\lambda I - A) = \{x \in X;\ Ax = \lambda x\}$ is called the *geometric multiplicity* of λ. The vector x is called a *generalized eigenvector* corresponding to the eigenvalue λ if $(\lambda I - A)^m x = 0$ for some $m \in \mathbb{N}$. The

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dimension of the space of generalized eigenvectors is called the *algebraic multiplicity* of the eigenvalue λ.

Theorem 1.1 is known in literature as the Riesz–Schauder–Fredholm theorem. (See, e.g., [82], p. 283.)

Theorem 1.1 *Let $T \in L(X, X)$ be a compact operator. Then its spectrum $\sigma(T)$ consists of an at most countable set of points of complex plane which has no point of accumulation except $\lambda = 0$. Moreover, every $\lambda \in \sigma(T)$ is eigenvalue of T of finite algebraic multiplicity.*

In particular, by Theorem 1.1 we have the following result.

Theorem 1.2 *Let A be a closed operator and densely defined operator in X with compact resolvent $(\lambda I - A)^{-1}$ for some $\lambda \in \rho(A)$. Then the spectrum $\sigma(T)$ consists of isolated eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ each of finite (algebraic) multiplicity m_j.*

If A is such an operator, then for each $N \in \mathbb{N}$, the spectrum $\sigma(A)$ can be written as

$$\sigma(A) = \{\lambda_j\}_{j=1}^{N} \cup \{\lambda_j\}_{j=1}^{N+1} \tag{1.1}$$

and if Γ is a closed curve in $\mathbb{C}$, which contains in interior $\{\lambda_j\}_{j=1}^{N}$, we set

$$P_N = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} d\lambda \tag{1.2}$$

and $X_N^1 = P_N X$, $X_N^2 = (I - P_N)X$. Then we have a decomposition of X in the direct sum

$$X = X_N^1 \oplus X_N^2, \qquad P_N^2 = P_N, \tag{1.3}$$

and if we set

$$A_N^1 = P_N A, \qquad A_N^2 = (I - P_N)A, \tag{1.4}$$

we have the following theorem (see Theorem 6.17 in [59]).

Theorem 1.3 *Under the assumptions of Theorem 1.2,*

$$A_N^i X_N^i \subset X_N^i, \quad i = 1, 2, \tag{1.5}$$

$$\sigma(A_N^i) = \{\lambda_j\}_{j=1}^{N}, \qquad \sigma(A_N^2) = \{\lambda_j\}_{j=N+1}^{\infty}. \tag{1.6}$$

If $N = 1$, then $\dim X_N^1 = m_1$ is just the algebraic multiplicity of the eigenvalue λ_1.

Definition 1.1 An eigenvalue λ of the operator A is called *semisimple* if the algebraic multiplicity of λ coincides with the geometric multiplicity.

In general, the algebraic multiplicity is greater than the geometric multiplicity.

We note that in finite dimension the spectrum of an operator consists of semi-simple eigenvalues if its Jordan form is diagonal.

If X is a Banach space, we denote by X^* its dual space endowed with the dual norm $\|x^*\|_{X^*} = \sup({}_X(x, x^*)_{X^*};\ \|x\|_X = 1\}$. (Here, ${}_X(x, x^*)_{X^*}$ is the value of x^* at x.)

If $A : X \to Y$ is a closed and densely defined operator (X, Y are Banach spaces), then the adjoint $A^* : Y^* \to X^*$ of A is defined by

$$\begin{aligned} &{}_{X^*}(A^*y^*, x)_X = {}_{Y^*}(y^*, Ax)_Y, \quad \forall x \in D(A), \\ &D(A^*) = \{y^* \in Y^*;\ \exists C > 0, |{}_{Y^*}(y^*, Ax)_Y| \le C\|x\|_X,\ \forall x \in D(A)\}. \end{aligned}$$

The adjoint operator A^* is closed, densely defined and $(\lambda I - A^*)^{-1} = ((\bar{\lambda} I - A)^{-1})^*$, $\forall \lambda \in \rho(A)$. Moreover, if λ is eigenvalue for A, then $\bar{\lambda}$ is eigenvalue for A^* of the same multiplicity.

If A is a closed and densely defined operator from X to X, its domain $D(A)$ is a Banach space with the norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$, $\forall x \in D(A)$, and we have $D(A) \subset X$ algebraically and topologically, that is, with dense and continuous embedding.

Assume now that $X = H$ is a Hilbert space with the norm $\|\cdot\|_H$ and scalar product $(\cdot, \cdot)_H$ and that there is $\lambda_0 \in \rho(A)$. Then, define the space $(D(A))'$ (the dual of $D(A)$ in the pairing $(\cdot, \cdot)$) as the completion of H in the norm

$$\|x\|_{(D(A))'} = \|(\lambda_0 I - A)^{-1} x\|_H, \quad \forall x \in H. \tag{1.7}$$

Then, we have

$$D(A) \subset H \subset (D(A))' \tag{1.8}$$

algebraically and topologically. Moreover, the operator $A : D(A) \subset H \to H$ has an extension denoted $\widetilde{A} : H \to (D(A^*))'$ defined by

$${}_{(D(A^*))'}(\widetilde{A}x, y)_{D(A^*)} = (x, A^*y), \quad \forall y \in D(A^*). \tag{1.9}$$

Of course, we have $\widetilde{A}x = Ax$, $\forall x \in D(A)$.

Moreover, since $\widetilde{A} : H \to (D(A^*))'$ is closed, by the closed graph theorem (see, e.g., [82], p. 77) we have that $\widetilde{A} \in L(H, (D(A^*))')$.

In applications to partial differential equations, the extension $\widetilde{A}$ of A incorporates into its domain $D(\widetilde{A}) = H$ boundary value conditions. (See an example in Sect. 1.2 below.)

1.2 Sobolev Spaces and Elliptic Boundary Value Problems

Throughout this section, until further notice, we assume that $\mathcal{O}$ is an open subset of R^d. To begin with, let us briefly recall the notion of *distribution*. Let $f = f(x)$ be a continuous complex-valued function on $\mathcal{O}$. By the *support* of f, abbreviated $\operatorname{supp} f$, we mean the closure of the set $\{x \in \mathcal{O};\ f(x) \neq 0\}$ or, equivalently, the smallest closed set of $\mathcal{O}$ outside of which f vanishes identically. We will denote by $C^k(\mathscr{O})$, $0 \le k \le \infty$, the set of all complex-valued functions defined in $\mathscr{O}$ which have continuous partial derivatives of order up to and including k (of any order $< \infty$ if $k = \infty$).

Let $C_0^k(\mathscr{O})$ denote the set of all functions $\varphi \in C^k(\mathscr{O})$ with compact support in $\mathscr{O}$.

We may introduce in $C_0^\infty(\mathscr{O})$ a convergence as follows. We say that the sequence $\{\varphi_k\} \subset C_0^\infty(\mathscr{O})$ is convergent to φ, denoted $\varphi_k \Rightarrow \varphi$, if

(a) There is a compact $K \subset \mathscr{O}$ such that $\operatorname{supp} \varphi_k \subset K$ for all $k = 1, \dots$.
(b) $\lim_{k\to\infty} D^\alpha \varphi_k = D^\alpha \varphi$ uniformly on K for all $\alpha = (\alpha_1, \dots, \alpha_n)$.

Here, $D^\alpha = D^{\alpha}_{x_1} \cdots D^{\alpha_n}_{x_N}$, $D_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. Equipped in this way, the space $C_0^\infty(\mathscr{O})$ is denoted by $\mathscr{D}(\mathscr{O})$. As a matter of fact, $\mathscr{D}(\mathscr{O})$ can be redefined as a locally convex, linear topological space with a suitable chosen family of seminorms.

A linear continuous functional u on $\mathscr{D}(\mathscr{O})$ is called a *distribution* on $\mathscr{O}$.

The set of all distributions on $\mathscr{O}$ is a linear space, denoted by $\mathscr{D}'(\mathscr{O})$.

The distribution is a natural extension of the notion of locally summable function on $\mathscr{O}$. Indeed, if $f \in L^1_{\mathrm{loc}}(\mathscr{O})$, then the linear functional u_f on $C_0^\infty(\mathscr{O})$ defined by

$$u_f(\varphi) = \int_{\mathscr{O}} f(x)\varphi(x)dx, \quad \forall \varphi \in C_0^\infty(\mathscr{O})$$

is a distribution on $\mathscr{O}$, that is, $u_f \in \mathscr{D}'(\mathscr{O})$.

Given $u \in \mathscr{D}'(\mathscr{O})$, by definition, the derivative of order $\alpha = (\alpha_1, \dots, \alpha_n)$, $D^\alpha u$, of u, is the distribution

$$(D^\alpha u)(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi), \quad \forall \varphi \in \mathscr{D}(\mathscr{O}), \text{ where } |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Let $\mathscr{O}$ be an open subset of R^d and let m be a positive integer. Denote by $H^m(\mathscr{O})$ the set of all real-valued functions $u \in L^2(\mathscr{O})$ such that distributional derivatives $D^\alpha u$ of u of order $|\alpha| \le m$ all belong to $L^2(\mathscr{O})$. In other words,

$$H^m(\mathscr{O}) = \{u \in L^2(\mathscr{O});\ D^\alpha u \in L^2(\mathscr{O}),\ |\alpha| \le m\}. \tag{1.10}$$

We present below a few basic properties of Sobolev spaces and refer to the books of Brezis [36], Adams [3], Barbu [11] for proofs.

Proposition 1.1 *$H^m(\mathscr{O})$ is a Hilbert space with the scalar product*

$$\langle u, v\rangle_m = \sum_{|\alpha|\le m} \int_{\mathscr{O}} D^\alpha u(x)\overline{D^\alpha v(x)}\,dx, \quad \forall u, v \in H^m(\mathscr{O}). \tag{1.11}$$

If $\mathscr{O} = (a, b)$, $-\infty < a < b < \infty$, $H^1(\mathscr{O})$ reduces to a subspace of absolutely continuous functions on the interval $[a, b]$.

More generally, for an integer $m \ge 1$ and $1 \le p \le \infty$, one defines the Sobolev space

$$W^{m,p}(\mathscr{O}) = \{u \in L^p(\mathscr{O});\ D^\alpha u \in L^p(\mathscr{O}),\ |\alpha| \le m\} \tag{1.12}$$

with the norm

$$\|u\|_{m,p} = \left(\sum_{|\alpha|\le m} \int_{\mathscr{O}} |D^\alpha u(x)|^p dx\right)^{1/p}. \tag{1.13}$$