V. I. ARNOLD · V. V. KOZLOV · A. I. NEISHTADT

Mathematical Aspects of Classical and Celestial Mechanics

3rd Edition

This work describes the fundamental principles, problems, and methods of classical mechanics. The main attention is devoted to the mathematical side of the subject. The authors have endeavored to give an exposition stressing the working apparatus of classical mechanics. The book is significantly expanded compared to the previous edition. The authors have added two chapters on the variational principles and methods of classical mechanics as well as on tensor invariants of equations of dynamics. Moreover, various other sections have been revised, added or expanded.

The main purpose of the book is to acquaint the reader with classical mechanics as a whole, in both its classical and its contemporary aspects. The book addresses all mathematicians, physicists and engineers.

From the reviews of the previous editions:

"... As an encyclopedia article, this book does not seek to serve as a textbook, nor to replace the original articles whose results it describes. The book's goal is to provide an overview, pointing out highlights and unsolved problems, and putting individual results into a coherent context. It is full of historical nuggets, many of them surprising. ... The examples are especially helpful; if a particular topic seems difficult, a later example frequently tames it. The writing is refreshingly direct, never degenerating into a vocabulary lesson for its own sake. The book accomplishes the goals it has set for itself. While it is not an introduction to the field, it is an excellent overview. ..." American Mathematical Monthly, Nov. 1989

"This is a book to curl up with in front of a fire on a cold winter's evening. ..." SIAM Reviews, Sept. 1989

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III

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Preface

In this book we describe the basic principles, problems, and methods of classical mechanics. Our main attention is devoted to the mathematical side of the subject. Although the physical background of the models considered here and the applied aspects of the phenomena studied in this book are explored to a considerably lesser extent, we have tried to set forth first and foremost the "working" apparatus of classical mechanics. This apparatus is contained mainly in Chapters 1, 3, 5, 6, and 8.

Chapter 1 is devoted to the basic mathematical models of classical mechanics that are usually used for describing the motion of real mechanical systems. Special attention is given to the study of motion with constraints and to the problems of realization of constraints in dynamics.

In Chapter 3 we discuss symmetry groups of mechanical systems and the corresponding conservation laws. We also expound various aspects of order-reduction theory for systems with symmetries, which is often used in applications.

Chapter 4 is devoted to variational principles and methods of classical mechanics. They allow one, in particular, to obtain non-trivial results on the existence of periodic trajectories. Special attention is given to the case where the region of possible motion has a non-empty boundary. Applications of the variational methods to the theory of stability of motion are indicated.

Chapter 5 contains a brief survey of the various approaches to the problem of integrability of the equations of motion and some of the most general and efficient methods of their integration. Diverse examples of integrated problems are given, which form the "golden reserve" of classical dynamics. The material of this chapter is used in Chapter 6, which is devoted to one of the most fruitful parts of mechanics – perturbation theory. The main task of perturbation theory is studying the problems of mechanics that are close to problems admitting exact integration. Elements of this theory (in particular, the well-known and widely used "averaging principle") arose in celestial mechanics in connection with attempts to take into account mutual gravitational perturbations of the planets of the Solar System. Adjoining Chapters 5 and 6 is Chapter 7, where the theoretical possibility of integrating the equations of motion (in a precisely defined sense) is studied. It turns out that integrable systems are a rare exception and this circumstance increases the importance of approximate integration methods expounded in Chapter 6. Chapter 2 is devoted to classical problems of celestial mechanics. It contains a description of the integrable two-body problem, the classification of final motions in the three-body problem, an analysis of collisions and regularization questions in the general problem of n gravitating points, and various limiting variants of this problem. The problems of celestial mechanics are discussed in Chapter 6 from the viewpoint of perturbation theory. Elements of the theory of oscillations of mechanical systems are presented in Chapter 8.

The last Chapter 9 is devoted to the tensor invariants of the equations of dynamics. These are tensor fields in the phase space that are invariant under the phase flow. They play an essential role both in the theory of exact integration of the equations of motion and in their qualitative analysis.

The book is significantly expanded by comparison with its previous editions (VINITI, 1985; Springer-Verlag, 1988, 1993, 1997). We have added Ch. 4 on variational principles and methods (§ 4.4.5 in it was written by S. V. Bolotin), Ch. 9 on the tensor invariants of equations of dynamics, § 2.7 of Ch. 2 on dynamics in spaces of constant curvature, §§ 6.1.10 and 6.4.7 of Ch. 6 on separatrix crossings, § 6.3.5 of Ch. 6 on diffusion without exponentially small effects (written by D. V. Treshchev), § 6.3.7 of Ch. 6 on KAM theory for lower-dimensional tori (written by M. B. Sevryuk), § 6.4.3 of Ch. 6 on adiabatic phases, § 7.6.3 of Ch. 7 on topological obstructions to integrability in the multidimensional case, § 7.6.4 of Ch. 7 on the ergodic properties of dynamical systems with multivalued Hamiltonians, and § 8.5.3 of Ch. 8 on the effect of gyroscopic forces on stability. We have substantially expanded § 6.1.7 of Ch. 6 on the effect of an isolated resonance, § 6.3.2 of Ch. 6 on invariant tori of the perturbed Hamiltonian system (with the participation of M. B. Sevryuk), § 6.3.4 of Ch. 6 on diffusion of slow variables (with the participation of S. V. Bolotin and D. V. Treshchev), § 7.2.1 on splitting of asymptotic surfaces conditions (with the participation of D. V. Treshchev). There are several other addenda. In this work we were greatly helped by S. V. Bolotin, M. B. Sevryuk, and D. V. Treshchev, to whom the authors are deeply grateful.

This English edition was prepared on the basis of the second Russian edition (Editorial URSS, 2002). The authors are deeply grateful to the translator E. I. Khukhro for fruitful collaboration.

Our text, of course, does not claim to be complete. Nor is it a textbook on theoretical mechanics: there are practically no detailed proofs in it. The main purpose of our work is to acquaint the reader with classical mechanics on the whole, both in its classical and most modern aspects. The reader can find the necessary proofs and more detailed information in the books and original research papers on this subject indicated at the end of this volume.

V. I. Arnold, V. V. Kozlov, A. I. Neishtadt

Contents

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Basic Principles of Classical Mechanics

For describing the motion of a mechanical system various mathematical models are used based on different "principles" – laws of motion. In this chapter we list the basic objects and principles of classical dynamics. The simplest and most important model of the motion of real bodies is Newtonian mechanics, which describes the motion of a free system of interacting points in three-dimensional Euclidean space. In § 1.6 we discuss the suitability of applying Newtonian mechanics when dealing with complicated models of motion.

1.1 Newtonian Mechanics

1.1.1 Space, Time, Motion

The space where the motion takes place is three-dimensional and Euclidean with a fixed orientation. We shall denote it by E^3. We fix some point $o \in E^3$ called the "origin of reference". Then the position of every point s in E^3 is uniquely determined by its position vector $\vec{os} = \mathbf{r}$ (whose initial point is o and end point is s). The set of all position vectors forms the three-dimensional vector space $\mathbb{R}^3$, which is equipped with the scalar product $\langle\ ,\ \rangle$.

Time is one-dimensional; it is denoted by t throughout. The set $\mathbb{R} = \{t\}$ is called the *time axis.*

A *motion* (or *path*) of the point s is a smooth map $\Delta \to E^3$, where Δ is an interval of the time axis. We say that the motion is defined on the interval Δ. If the origin (point o) is fixed, then every motion is uniquely determined by a smooth vector-function $\mathbf{r}\colon\ \Delta \to \mathbb{R}^3$.

The image of the interval Δ under the map $t \mapsto \mathbf{r}(t)$ is called the *trajectory* or *orbit of the point s.*

The *velocity* $\mathbf{v}$ of the point s at an instant $t \in \Delta$ is by definition the derivative $d\mathbf{r}/dt = \dot{\mathbf{r}}(t) \in \mathbb{R}^3$. Clearly the velocity is independent of the choice of the origin.