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The Geometry of Infinite-Dimensional Groups

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The Geometry of Infinite-Dimensional Groups

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To our Teachers:

to Vladimir Igorevich Arnold
and to the memory of Peter Slodowy

Preface

The aim of this monograph is to give an overview of various classes of infinite-dimensional Lie groups and their applications, mostly in Hamiltonian mechanics, fluid dynamics, integrable systems, and complex geometry. We have chosen to present the unifying ideas of the theory by concentrating on specific types and examples of infinite-dimensional Lie groups. Of course, the selection of the topics is largely influenced by the taste of the authors, but we hope that this selection is wide enough to describe various phenomena arising in the geometry of infinite-dimensional Lie groups and to convince the reader that they are appealing objects to study from both purely mathematical and more applied points of view. This book can be thought of as complementary to the existing more algebraic treatments, in particular, those covering the structure and representation theory of infinite-dimensional Lie algebras, as well as to more analytic ones developing calculus on infinite-dimensional manifolds.

This monograph originated from advanced graduate courses and mini-courses on infinite-dimensional groups and gauge theory given by the first author at the University of Toronto, at the CIRM in Marseille, and at the Ecole Polytechnique in Paris in 2001–2004. It is based on various classical and recent results that have shaped this newly emerged part of infinite-dimensional geometry and group theory.

Our intention was to make the book concise, relatively self-contained, and useful in a graduate course. For this reason, throughout the text, we have included a large number of problems, ranging from simple exercises to open questions. At the end of each section we provide bibliographical notes, trying to make the literature guide more comprehensive, in an attempt to bring the interested reader in contact with some of the most recent developments in this exciting subject, the geometry of infinite-dimensional groups. We hope that this book will be useful to both students and researchers in Lie theory, geometry, and Hamiltonian systems.

It is our pleasure to thank all those who helped us with the preparation of this manuscript. We are deeply indebted to our teachers, collaborators, and friends, who influenced our view of the subject: V. Arnold, Ya. Brenier, H. Bursztyn, Ya. Eliashberg, P. Etingof, V. Fock, I. Frenkel, D. Fuchs, A. Kirillov, F. Malikov, G. Misiołek, R. Moraru, N. Nekrasov, V. Ovsienko, C. Roger, A. Rosly, V. Rubtsov, A. Schwarz, G. Segal, M. Semenov-Tian-Shansky, A. Shnirelman, P. Slodowy, S. Tabachnikov, A. Todorov, A. Veselov, F. Wagemann, J. Weitsman, I. Zakharevich, and many others. We are particularly grateful to Alexei Rosly, the joint projects with whom inspired a large part, in particular the "application chapter," of this book, and who made numerous invaluable remarks on the manuscript. We thank the participants of the graduate courses for their stimulating questions and remarks. Our special thanks go to M. Peters and the Springer team for their invariable help and to D. Kramer for careful editing of the text.

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Finally, we thank our families (kids included!) for their tireless moral support and encouragement throughout the over-stretched work on the manuscript.

Contents

Introduction

What is a group? Algebraists teach that this is supposedly a set with two operations that satisfy a load of easily-forgettable axioms...

V.I. Arnold "On teaching mathematics" [20]

Today one cannot imagine mathematics and physics without Lie groups, which lie at the foundation of so many structures and theories. Many of these groups are of infinite dimension and they arise naturally in problems related to differential and algebraic geometry, knot theory, fluid dynamics, cosmology, and string theory. Such groups often appear as symmetries of various evolution equations, and their applications range from quantum mechanics to meteorology. Although infinite-dimensional Lie groups have been investigated for quite some time, the scope of applicability of a general theory of such groups is still rather limited. The main reason for this is that infinite-dimensional Lie groups exhibit very peculiar features.

Let us look at the relation between a Lie group and its Lie algebra as an example. As is well known, in finite dimensions each Lie group is, at least locally near the identity, completely described by its Lie algebra. This is achieved with the help of the exponential map, which is a local diffeomorphism from the Lie algebra to the Lie group itself. In infinite dimensions, this correspondence is no longer so straightforward. There may exist Lie groups that do not admit an exponential map. Furthermore, even if the exponential map exists for a given group, it may not be a local diffeomorphism. Another pathology in infinite dimensions is the failure of Lie's third theorem, stating that every finite-dimensional Lie algebra is the Lie algebra attached to some finite-dimensional Lie group. In contrast, there exist infinite-dimensional Lie algebras that do not correspond to any Lie group at all.

In order to avoid such pathologies, any version of a general theory of infinite-dimensional Lie groups would have to restrict its attention to certain classes of such groups and study them separately. For example, one might consider the class of Banach Lie groups, i.e., Lie groups that are locally modeled on Banach spaces and behave very much like finite-dimensional Lie groups. For Banach Lie groups the exponential map always exists and is a local diffeomorphism. However, restricting to Banach Lie groups would already exclude the important case of diffeomorphism groups, and so on. This is why the attempts to develop a unified theory of infinite-dimensional differential geometry, and hence, of infinite-dimensional Lie groups, are still far from reaching greater generality.

In the present book, we choose a different approach. Instead of trying to develop a general theory of such groups, we concentrate on various examples of infinite-dimensional Lie groups, which lead to a realm of important applications.

The examples we treat here mainly belong to three general types of infinite-dimensional Lie groups: groups of diffeomorphisms, gauge transformation groups, and groups of pseudodifferential operators. There are numerous interrelations between various groups appearing in this book. For example, the group of diffeomorphisms of a compact manifold acts naturally on the group of currents over this manifold. When this manifold is a circle, this action gives rise to a deep connection between the representation theory of the Virasoro algebra and the Kac–Moody algebras. In the geometric setting of this book, this relation manifests itself in the correspondence between the coadjoint orbits of these groups.

Another strand connecting various groups considered below is the theme of the "ladder" of current groups. We regard the passage from finite-dimensional Lie groups (i.e., "current groups at a point") to loop groups (i.e., current groups on the circle), and then to double loop groups (current groups on the two-dimensional torus) as a "ladder of groups." On the side of dynamical systems this is revealed in the passage from rational to trigonometric and to elliptic Calogero–Moser systems. The passage from ordinary loop groups to double loop groups also serves as the starting point of a "real–complex correspondence" discussed in the chapter on applications of groups. There we study moduli spaces of flat or integrable connections on real and complex surfaces using the geometry of coadjoint orbits of these two types of groups.

Most of main objects studied in the book can be summarized in the table below.

In Chapter II, in a sense, we are moving horizontally, along the first row of this table. We study affine and elliptic groups, their orbits and geometry, as well as the related Calogero–Moser systems. We also describe in this chapter many Lie groups and Lie algebras outside the scope of this table: groups of diffeomorphisms, the Virasoro group, groups of pseudodifferential operators. In the appendices one can find the Krichever–Novikov algebras, $\mathfrak{gl}_\infty$, and other related objects.

In Chapter III we move vertically in this table and mostly focus on the current groups and on their parallel description in topological and holomorphic contexts. While affine and elliptic Lie groups correspond to the base dimension

Base dimension	Real / topological theory	Complex / holomorphic theory
1	affine (or, loop) groups (orbits $\sim$ monodromies over a circle)	elliptic (or, double loop) groups (orbits $\sim$ holomorphic bundles over an elliptic curve)
2	flat connections over a Riemann surface (Poisson structures)	holomorphic bundles over a complex surface (holomorphic Poisson structures)
3	connections over a threefold (Chern–Simons functional, singular homology, classical linking)	partial connections over a complex threefold (holomorphic Chern–Simons functional, polar homology, holomorphic linking)

1, either real or complex, in dimension 2 we describe the spaces of connections on real or complex surfaces, as well as the symplectic and Poisson structures on the corresponding moduli spaces. (In the table the main focus of study is mentioned in the parentheses of the corresponding block.) In dimension 3 the study of the Chern–Simons functional and its holomorphic version leads one to the notions of classical and holomorphic linking, and to the corresponding homology theories. (Although we confined ourselves to three dimensions, one can continue this table to dimension 4 and higher, which brings in the Yang–Mills and many other interesting functionals; see, e.g., [85].)

Note that the objects (groups, connections, etc.) in each row of this table usually dictate the structure of objects in the row above it, although the "interaction of the rows" is different in the real and complex cases. Namely, in the real setting, the lower-dimensional manifolds appear as the boundary of real manifolds of one dimension higher. For the complex case, the low-dimensional complex varieties arise as divisors in higher-dimensional ones; see details in Chapter III.

Overview of the content. Here are several details on the contents of various chapters and sections.

In Chapter I, we recall some notions and facts from Lie theory and symplectic geometry used throughout the book. Starting with the definition of a Lie group, we review the main related concepts of its Lie algebra, the adjoint and coadjoint representations, and introduce central extensions of Lie groups and algebras. We then recall some notions from symplectic geometry, including Arnold's formulation of the Euler equations on a Lie group, which are the equations for the geodesic flow with respect to a one-sided invariant metric on the group. This setting allows one to describe on the same footing many finite- and infinite-dimensional dynamical systems, including the classical Euler equations for both a rigid body and an ideal fluid, the Korteweg–de Vries equation, and the equations of magnetohydrodynamics. Finally, the preliminaries cover the Marsden–Weinstein Hamiltonian reduction, a method often used to describe complicated Hamiltonian systems starting with a simple one on a nonreduced space, by "dividing out" extra symmetries of the system.

Chapter II is the main part of this book, and can be viewed as a walk through the zoo of the various types of infinite-dimensional Lie groups. We tried to describe these groups by presenting their definitions, possible explicit constructions, information on (or, in some cases, even the complete classification of) their coadjoint orbits. We also discuss relations of these groups to various Hamiltonian systems, elaborating, whenever possible, on important constructions related to integrability of such systems. The table of contents is rather self-explanatory.

We start this chapter by introducing the loop group of a compact Lie group, one of the most studied types of infinite-dimensional groups. In Section 1, we construct its universal central extension, the corresponding Lie algebra (called the affine Kac–Moody Lie algebra), and classify the corresponding coadjoint orbits. We also return to discuss the relation of this Lie algebra to the Landau–Lifschitz equation and the Calogero–Moser integrable system in the later sections.

In Section 2 we turn to the group of diffeomorphisms of the circle and its Lie algebra of smooth vector fields. Both the group and the Lie algebra admit universal central extensions, called the Virasoro–Bott group and the Virasoro algebra respectively. It turns out that the coadjoint orbits of the Virasoro–Bott group can be classified in a manner similar to that for the orbits of the loop groups. The Euler equation for a natural right-invariant metric on the Virasoro–Bott group is the famous Korteweg–de Vries (KdV) equation, which describes waves in shallow water. Furthermore, the Euler nature of the KdV helps one to show that this equation is completely integrable.

Section 3 is devoted to various diffeomorphism groups and, in particular, to the group of volume-preserving diffeomorphisms of a compact Riemannian manifold M. The Euler equations on this group are the Euler equations for an ideal incompressible fluid filling M. Enlarging the group of volume-preserving diffeomorphisms by either smooth functions or vector fields on M gives the Euler equations of gas dynamics or of magnetohydrodynamics, respectively. We also mention some results on the Riemannian geometry of diffeomorphism groups and discuss the relation of the latter to the the Marsden–Weinstein symplectic structure on the space of immersed curves in $\mathbb{R}^3$.

Section 4 deals with the group of pseudodifferential symbols (or operators) on the circle. It turns out that this group can be endowed with the structure of a Poisson Lie group, where the corresponding Poisson structures are given by the Adler–Gelfand–Dickey brackets. The dynamical systems naturally corresponding to this group are the Kadomtsev–Petviashvili hierarchy, the higher n-KdV equations, and the nonlinear Schrödinger equation.

Section 5 returns to the loop groups "at the next level": here we deal with their generalizations, elliptic Lie groups and the corresponding Lie algebras. These groups are extensions of the groups of double loops, i.e., the groups of smooth maps from a two-dimensional torus to a finite-dimensional complex