VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions,
Celestial Mechanics, and KAM Theory
1957–1965

Springer

VLADIMIR I. ARNOLD

Collected Works

Springer

Vladimir I. Arnold, 1961
Photograph by Jürgen Moser

VLADIMIR I. ARNOLD

Collected Works

VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions, Celestial Mechanics
and KAM Theory, 1957–1965

VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions, Celestial Mechanics
and KAM Theory, 1957–1965

Edited by

Alexander B. Givental
Boris A. Khesin
Jerrold E. Marsden
Alexander N. Varchenko
Victor A. Vassilev
Oleg Ya. Viro
Vladimir M. Zakalyukin

Springer

Vladimir I. Arnold

Russian Academy of Sciences
Steklov Mathematical Institute
ul. Gubkina 8
Moscow 117966
Russia

Editors

A.B. Givental, B.A. Khesin, J.E. Marsden, A.N. Varchenko,
V.A. Vassilev, O.Ya. Viro, V.M. Zakalyukin

ISBN 978-3-642-01741-4 e-ISBN 978-3-642-01742-1
DOI 10.1007/978-3-642-01742-1
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009931933

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Vladimir Igorevich Arnold is one of the most influential mathematicians of our time. V.I. Arnold launched several mathematical domains (such as modern geometric mechanics, symplectic topology, and topological fluid dynamics) and contributed, in a fundamental way, to the foundations and methods in many subjects, from ordinary differential equations and celestial mechanics to singularity theory and real algebraic geometry. Even a quick look at a partial list of notions named after Arnold already gives an overview of the variety of such theories and domains:

KAM (Kolmogorov–Arnold–Moser) theory,
The Arnold conjectures in symplectic topology,
The Hilbert–Arnold problem for the number of zeros of abelian integrals,
Arnold's inequality, comparison, and complexification method in real algebraic geometry,
Arnold–Kolmogorov solution of Hilbert's 13th problem,
Arnold's spectral sequence in singularity theory,
Arnold diffusion,
The Euler–Poincaré–Arnold equations for geodesics on Lie groups,
Arnold's stability criterion in hydrodynamics,
ABC (Arnold–Beltrami–Childress) flows in fluid dynamics,
The Arnold–Korkina dynamo,
Arnold's cat map,
The Arnold–Liouville theorem in integrable systems,
Arnold's continued fractions,
Arnold's interpretation of the Maslov index,
Arnold's relation in cohomology of braid groups,
Arnold tongues in bifurcation theory,
The Jordan–Arnold normal forms for families of matrices,
The Arnold invariants of plane curves.

Arnold wrote some 700 papers, and many books, including 10 university textbooks. He is known for his lucid writing style, which combines mathematical rigour with physical and geometric intuition. Arnold's books on *Ordinar*y *differential equation*s and *Mathematical method*s *of classical mechanics* became mathematical bestsellers and integral parts of the mathematical education of students throughout the world.

Some Comments on V.I. Arnold's Biography and Distinctions

V.I. Arnold was born on June 12, 1937 in Odessa, USSR. In 1954–1959 he was a student at the Department of Mechanics and Mathematics, Moscow State University. His M.Sc. Diploma work was entitled "On mappings of a circle to itself." The degree of a "candidate of physical-mathematical sciences" was conferred to him in 1961 by the Keldysh Applied Mathematics Institute, Moscow, and his thesis advisor was A.N. Kolmogorov. The thesis described the representation of continuous functions of three variables as superpositions of continuous functions of two variables, thus completing the solution of Hilbert's 13th problem. Arnold obtained this result back in 1957, being a third year undergraduate student. By then A.N. Kolmogorov showed that continuous functions of more variables can be represented as superpositions of continuous functions of three variables. The degree of a "doctor of physical-mathematical sciences" was awarded to him in 1963 by the same Institute for Arnold's thesis on the stability of Hamiltonian systems, which became a part of what is now known as KAM theory.

After graduating from Moscow State University in 1959, Arnold worked there until 1986 and then at the Steklov Mathematical Institute and the University of Paris IX.

Arnold became a member of the USSR Academy of Sciences in 1986. He is an Honorary member of the London Mathematical Society (1976), a member of the French Academy of Science (1983), the National Academy of Sciences, USA (1984), the American Academy of Arts and Sciences, USA (1987), the Royal Society of London (1988), Academia Lincei Roma (1988), the American Philosophical Society (1989), the Russian Academy of Natural Sciences (1991). Arnold served as a vice-president of the International Union of Mathematicians in 1999–2003.

Arnold has been a recipient of many awards among which are the Lenin Prize (1965, with Andrey Kolmogorov), the Crafoord Prize (1982, with Louis Nirenberg), the Lobachevsky Prize of Russian Academy of Sciences (1992), the Harvey prize (1994), the Dannie Heineman Prize for Mathematical Physics (2001), the Wolf Prize in Mathematics (2001), the State Prize of the Russian Federation (2007), and the Shaw Prize in mathematical sciences (2008).

One of the most unusual distinctions is that there is a small planet Vladarnolda, discovered in 1981 and registered under #10031, named after Vladimir Arnold. As of 2006 Arnold was reported to have the highest citation index among Russian scientists.

In one of his interviews V.I. Arnold said: "The evolution of mathematics resembles the fast revolution of a wheel, so that drops of water fly off in all directions. Current fashion resembles the streams that leave the main trajectory in tangential directions. These streams of works of imitation are the most noticeable since they constitute the main part of the total volume, but they die out soon after departing the wheel. To stay on the wheel, one must apply effort in the direction perpendicular to the main flow."

With this volume Springer starts an ongoing project of putting together Arnold's work since his very first papers (not including Arnold's books.) Arnold continues to do research and write mathematics at an enviable pace. From an originally planned 8 volume edition of his Collected Works, we already have to increase this estimate to 10 volumes, and there may be more. The papers are organized chronologically. One might regard this as an attempt to trace to some extent the evolution of the interests of V.I. Arnold and cross-fertilization of his ideas. They are presented using the original English translations, whenever such were available. Although Arnold's works are very diverse in terms of subjects, we group each volume around particular topics, mainly occupying Arnold's attention during the corresponding period.

Volume I covers the years 1957 to 1965 and is devoted mostly to the representations of functions, celestial mechanics, and to what is today known as the KAM theory.

Acknowledgements. The Editors thank the Göttingen State and University Library and the Caltech library for providing the article originals for this edition. They also thank the Springer office in Heidelberg for its multilateral help and making this huge project of the Collected Works a reality.

March 2009

Alexander Givental
Boris Khesin
Jerrold Marsden
Alexander Varchenko
Victor Vassiliev
Oleg Viro
Vladimir Zakalyukin

Contents

ON THE REPRESENTATION OF FUNCTIONS OF TWO VARIABLES IN THE FORM $\chi[\phi(x) + \psi(y)]$*

V.I. Arnol'd
translated by Gerald Gould

1. Kolmogorov proved [1] that the set of functions of two variables representable as a certain combination of continuous functions of one variable and addition is everywhere dense in the space $C(E^2)$ of continuous functions defined on the square E^2. It follows immediately from our result proved below that this is not true for the simplest combinations: the set of functions of the form $\chi[\phi(x) + \psi(y)]$ even turns out to be nowhere dense in $C(E^2)$.

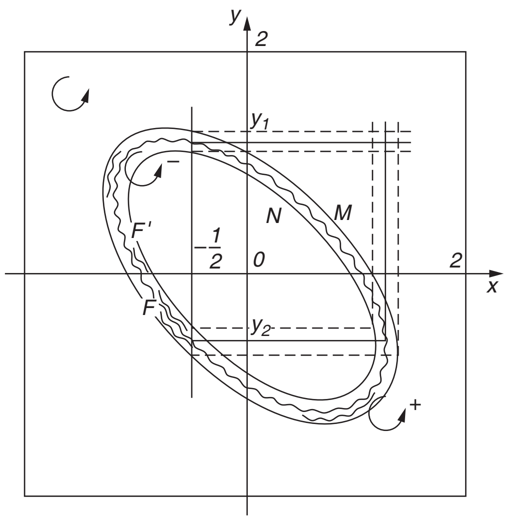

Fig. 1.

We shall indicate *a closed subset N of the square $|x| \leqslant 2$, $|y| \leqslant 2$ (Fig. 1) such that for any continuous function $f(x, y)$ vanishing on (and only on) N there exists $\delta(f) > 0$ such that $|f(x, y) - \chi[\phi(x) + \psi(y)]| \geqslant \delta$ at some point of this square for any continuous functions χ, ϕ and ψ; every function having*

* Uspekhi Math. Nauk **12**, No. 2, 119–121 (1957)

N as its level set is 'with a neighbourhood' non-representable in the form $\chi[\phi(x)+\psi(y)]$ *. An example of such a set N is the ellipse* $(x+y)^2+\frac{(x-y)^2}{4}=1$.

We shall prove this. Since $f(x,y)$ is of constant sign outside the ellipse we can assume that $f(x,y)>0$ there. Then clearly there exists $\delta>0$ such that $f(x,y)>2\delta$ at all points in the region $G \stackrel{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}>\frac{5}{4}$, that is, outside the ellipse $M \stackrel{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}=\frac{5}{4}$. Suppose that there exist continuous functions $\phi(x)$, $\psi(y)$, $\chi(z)$ such that $|f(x,y)-\chi[\phi(x)-\psi(y)]|<\delta$ † for all (x,y), $2\leqslant x, y\leqslant 2$. Then the inequality $\chi[\phi(x)+\psi(y)]<\delta$ holds on N and the inequality $\chi[\phi(x)+\psi(y)]>\delta$ holds on M.

The largest open connected sets $G^-\supset N$ and $G^+\supset G$,⋆ where $\chi[\phi(x)+\psi(y)]<\delta$ and $\chi[\phi(x)+\psi(y)]>\delta$, respectively, are separated by the closed set F where $\chi[\phi(x)+\psi(y)]=\delta$ (that is, each continuum intersecting G^- and G^+ also intersects F), because the continuous function $\chi[\phi(x)+\psi(y)]$ on a continuum takes all values between any two given values. By a well-known theorem (Theorem E in [2]) the boundary of G^+ has a component $F'\subseteq F$ already separating G^- and G^+, and hence M and N. We claim that the continuous function $\phi(x)+\psi(y)$ is constant on F'. Indeed, suppose that, on the contrary, $z_1=\phi(x)+\psi(y)|_a<\phi(x)+\psi(y)|_b=z_2$, where $a,b\in F'$. Then in a sufficiently small neighbourhood of a there is a point $a'\in G^+$ where $\phi(x)+\psi(y)<z_1+\frac{z_2-z_1}{3}$, and in a sufficiently neighbourhood of b there is a point $b'\in G^+$ where $\phi(x)+\psi(y)>z_2-\frac{z_2-z_1}{3}$. Therefore on a polygonal line joining a' and b' in G^+ there is a point c where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$; also there is a point c on the continuum F' where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$. Consequently, $\chi[\phi(x)+\psi(y)]|_{c'}=\chi[\phi(x)+\psi(y)]|_c$, which contradicts the conditions $c'\in G^+$, $c\in F'$.

We denote by z the unique value of $\phi(x)+\psi(y)$ at points of F'. Then on the intervals $x=-\frac{1}{2}$, $y\in[1.1,1.22]$ and $x=-\frac{1}{2}$, $y\in[-0.62,-0.5]$ intersecting M and N there are points $(-\frac{1}{2},y_1)$ and $(-\frac{1}{2},y_2)$ at which $\phi(x)+\psi(y)=z$. There is such a point (x_1,y_2) on the interval on which the line $y=y_2$ intersects the strip between M and N for $x>0$.

It follows from the equalities⋆⋆

$$\phi(-\tfrac{1}{2})+\psi(y_1)=z\,,$$
$$\phi(-\tfrac{1}{2})+\phi(y_2)=z\,,$$
$$\phi(x_1)+\psi(y_2)=z$$

that $\phi(x_1)+\psi(y_1)=z$ and $\chi[\phi(x_1)+\psi(y_2)]=\delta$. However, it is easy to see that the point (x_1,y_1) lies in G, therefore $\chi[\phi(x_1)+\psi(y_2)]>\delta$. This contradiction proves the 'stable' non-representability of $f(x,y)$ in the form $\chi[\phi(x)+\psi(y)]$;

† *Translator's note*: This should be $|f(x,y)-\chi[\phi(x)+\psi(y)]|<\delta$.

⋆ *Translator's note*: This should be $G^+\supset M$.

⋆⋆ *Translator's note*: The second of these inequalities contains a misprint. It should read $\phi(-\frac{1}{2})+\psi(y_2)=z$.