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Dependence, Risk Bounds, Optimal Allocations and Portfolios

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Preface

This book gives an introduction to basic concepts and methods in mathematical risk analysis, in particular to those parts of risk theory which are of particular relevance in finance and insurance. The description of the influence of dependence in multivariate stochastic models for risk vectors is the main focus of the text.

In the first part we introduce basic probabilistic tools and methods of distributional analysis and describe their use in the modelling of dependence and to derive risk bounds in these models. The second part of this book is concerned with measures of risk with a particular view towards risk measures in financial and insurance context. The final parts are devoted to relevant applications such as optimal risk allocations problems and optimal portfolio problems. Throughout we give short accounts of the basic methods used from stochastic ordering, from duality theory, from extreme value theory, convex analysis, and empirical processes as they are needed in the applications.

The focus is on presentation of the main ideas and methods and on their relevance for practical application. In favour of better readability we give only proofs of those results which do not need too much preparation or which are not too extended. The text is mostly self-contained. We make use of several basic results from stochastic ordering and distributional analysis which can be spotted easily in related textbooks.

The book can be used as a textbook for an advanced undergraduate third year course or for a graduate level mathematical risk analysis course. Good knowledge of basic probability and statistics as well as of general mathematics is a prerequisite. Some more advanced mathematical methods are explained on a non-technical level. The book should also be of interest as a reference text giving a clearly structured and up-to-date treatment of the main concepts and techniques used in this area. It also gives a guide to actual research; it points out relevant present research topics and describes the state of the art. In particular its aim is to give orientation to an interested researcher entering the field and moreover to help to acquire a solid fundament for working in this area.

The notion of dependence raises several basic issues. The first one is the construction of stochastic models of dependent risk vectors broad enough to describe all relevant classes of dependent risks. It is pointed out that fundamental tools for this purpose are the distributional transform, the quantile transform, and their multivariate extensions. These tools give an easy access to the Fréchet class, which is a synonym for the class of all possible dependence models. In particular a simple application of these distributional transforms gives the general form of Sklar's representation theorem and thus the notion of copula. The multivariate quantile transform yields a construction method for random vectors with specified general distribution and is a basic tool for simulation. The multivariate distributional transform on the other side, transforms a random vector to a vector with iid uniformly distributed components. This transform extends a classical result of Rosenblatt (1952) and has some important applications to goodness of fit tests and to identification procedures. Some concrete classes of constructions of multivariate copula models are described by methods like L^2-projections or the pair-copula constructions.

A classical topic in the analysis of risk is the development of sharp risk bounds in dependence models. The historical origins of this question are the Hoeffding–Fréchet bounds which give sharp upper and lower bounds for the covariance and the joint distribution function of two random variables X, Y with distribution functions F, G. These results have been extended to the problem of establishing sharp bounds for general risk functionals w.r.t. Fréchet classes. Important progress on this class of problems was obtained by the development of a corresponding duality theory, which was motivated by this problem of getting bounds for dependence functionals. It turned out that this duality theory in case of a two-fold product space connects up with the Monge–Kantorovich mass-transportation theory which aimed to describe minimal distances or transport costs between two distributions. By means of duality theory several basic sharp dependence bounds could be determined.

As a consequence the notion of comonotonicity is identified as the worst case dependence structure, in case the components of the portfolio are real. These findings were further extended by means of various stochastic ordering results concerning diffusion type orderings (as convex order or stop-loss order) and also concerning dependence orderings (like super-modular or directionally-convex ordering). W.r.t. all convex law invariant risk measures comonotonicity is the worst case dependence structure of the joint portfolio.

An exposition of the representation theory of convex risk measures and its basic properties like continuity properties is given on spaces of L^p-risks. Also extensions to risk measures on portfolio vectors are detailed. These extensions allow one to include for optimal allocation or portfolio problems the important aspect of dependence within the portfolio components. A fundamental question concerning the dependence structure is on the existence and form of a worst case dependence structure – generalizing comonotonicity – for a sample $X_1, \dots, X_n$ of portfolio vectors. It turns out however that a universally worst case dependence structure does not exist any more in higher dimension. But it is possible to describe worst case dependent portfolios w.r.t. specific multivariate risk measures. Here again a close connection with mass transportation comes into play. The max-correlation risk measures which are defined via mass transportation problems are the building blocks of all law invariant risk measures and thus take the role of the "average value at risk" risk measure in one dimension. Worst case dependence structures then are identified by comonotonicity w.r.t. worst case scenario measures.

In the final two chapters some relevant classes of optimal risk allocation and portfolio optimization problems are dealt with. The risk allocation problems are closely connected with optimal investment problems or minimal demand problems in finance and insurance. We also discuss classical and recent results on optimal (re-)insurance contracts. By combination of stochastic ordering results and results on worst case risk measures some simplified derivation of these optimality results can be given. Optimal portfolios are determined also from the point of view to minimize the sensibility to extremal risk events. These results are based on extreme value theory and supplement the usual finite risk analysis given by extensions of the classical Markowitz theory to the frame of risk measures. The notion of asymptotic portfolio loss order allows us to compare in this respect different stochastic loss models.

The aim of the book is to present relevant methods and tools to deal with the influence of dependence on various problems of risk analysis. It also discusses in detail some relevant applications to optimal risk allocation and optimal portfolio problems in finance and in insurance. The content of this book represents areas of my research over the last 20–30 years. The work on dependence, stochastic ordering, and risk bounds has been combined in more recent years with the new developments on risk measures and related optimization problems. As a result the book is not an encyclopaedic presentation. Several relevant subjects of mathematical risk analysis are not dealt with in this book. The basis of this book are several of my survey papers and oral presentations on dependence, risk bounds, and stochastic orderings, as dealt with in this book. Of particular mention are surveys on the theory of Fréchet bounds, which were presented and published in the series of volumes of the conferences on *Probabilities with given marginals* started by the Rome 1990 conference. The main topics of stochastic orderings in particular dependence orderings as described in this text, the related duality theory, and the distributional transforms go back to my habilitation thesis in 1979 and some related publications in the following years.

The basis and motivation for working in the area of risk measures naturally arose from the fundamental work of Delbaen (2000) and Föllmer and Schied (2011). My particular interest in this area was to combine this theory with the analysis of dependence properties. This aim has also been followed with more focus on applications in risk management in the book of McNeil et al. (2005b) giving a rich source of techniques and methods. In the book of Pflug and Römisch (2007) this theory is combined with statistical and decision theoretic concepts. My more recent interest was also driven by insurance applications and problems of optimal insurance contracts as presented in Kaas et al. (2001) and in more detail in the more recent book of Denuit et al. (2005) which is also focused on the role of stochastic ordering, risk measures, and dependence in insurance. In that book a much broader exposition of basics in these areas is given and applied insurance problems are included in much more detail.

Finally, I would like to express my gratitude and appreciation to all those whom I had the pleasure to work with over the years or who gave some inspiration to the work, either personally or by their work.

Fruitful cooperation with Svetlozar T. Rachev in the 1990s on mass transportation problems and also with Norbert Gaffke, Juan Cuesta-Albertos, Doraiswany Ramachandran, and Michael Ludkovski, enriched my view on the area of considered risk problems over a period of more than 10 years. The part on risk measures and newer developments on risk bounds and optimal allocation and portfolio problems reflects also a lot of work and cooperation with several of my former students. I would like to mention Ludger Uckelmann, Thomas Goll, Maike Kaina, Irina Weber, Christian Burgert, Jan Bergenthum, Swen Kiesel, Victor Wolf, and Georg Mainik. I must also mention recent joint work with Giovanni Puccetti and Paul Embrechts on extended risk bounds.

An essential impetus to start this work came from Damir Filipovic (who suggested I undertake this project). However it took two years to plan this book and for it to come to life based on a sabbatical. Particular thanks are due to Monika Hattenbach for the excellent typing and organizational work. She was supported in some parts by Thomas Lais. Many thanks also to Thomas Mikosch for his comments on parts of the manuscript, to several reviewers for their mostly friendly and encouraging comments and to Catriona Byrne for her kind and competent guidance through the problems of choosing the right book series and her careful organization of the review and production process.

Freiburg, Germany Ludger Rüschendorf
December 2012

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Part I
Stochastic Dependence and Extremal Risk

In financial and insurance risk management as in many further areas of probabilistic modelling as for example in network analysis there are typically several sources of risks. In insurance there are the different contracts held by an insurance company, in finance the portfolio held by a bank is composed of many risky assets. The various risk components are typically not independent of each other. In consequence it became necessary to develop the proper tools to describe the relevant statistical models for dependent variables and to analyse their properties.

There are two basic tasks to face this situation. The first one is the problem of "dependence modelling". Here we are given a vector of risks $X = (X_1, \dots, X_n)$ where the components X_i have known distributions P_i. X_i themselves might be one-dimensional or higher dimensional risks. The joint distribution P^X of X then is an element of the "Fréchet class" $\mathcal{M}(P_1, \dots, P_n)$ of all probability measures P on the product space which have marginals P_i, i.e. $P^{\pi_i} = P_i$, $1 \leq i \leq n$, where π_i are the projections on the i-th component. To describe models for the dependence structure of X is equivalent to describe (parametric) submodels of the Fréchet class $\mathcal{M}(P_1, \dots, P_n)$. In the case of one-dimensional marginals with distribution functions F_i also the notion $\mathcal{F}(F_1, \dots, F_n)$ is used for the Fréchet class of corresponding distribution functions. The basic notion of copula introduced by Sklar aims to separate the description of the dependence part of a distribution and the marginal part in the case of one-dimensional marginals. In consequence dependence models are given by specifying the marginals and specifying the corresponding copula models.

The second main subject of describing "dependence" are notions of dependence orderings, corresponding dependence measures and the description of bounds on the possible influence of dependence on certain risk functionals like the risk of the joint portfolio in finance or in insurance. Basic results in this direction are the classical Hoeffding–Fréchet bounds which specify upper and lower bounds for the joint distribution function $F = F_X$ of the risk vector X. In this connection the notion of "comonotonicity" is used to describe the worst case dependence structure for one-dimensional marginal risks.